

The Sturm Property of Coordinator Polynomials of Type D

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Abstract

It is known that the coordinator polynomials $h_{A_n}(x)$ of type A form a Sturm sequence since they can be expressed in terms of the Legendre polynomials. In this paper, we show that the coordinator polynomials $h_{D_n}(x)$ of type D form a Sturm sequence. Our proof is based on the technique of substituting the variable x by a trigonometric function. The same method applies to the real-rootedness of the coordinator polynomials $h_{C_n}(x)$ of type C .

Keywords: Coordinator polynomial, Sturm sequence, Trigonometric substitution.

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1 Introduction

Polynomials arising from combinatorics are often real-rooted. Basic examples include the generating functions of binomial coefficients, of Stirling numbers of the first kind and the second kind, and of Eulerian numbers, see, for example, Brenti [6, 7], Pitman [13], Stanley [16], Wang and Yeh [18].

Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree n with nonnegative coefficients. We say that $f(x)$ is real-rooted if all its roots are real. There is a characterization of real-rooted polynomials in the theory of total positivity, see Karlin [10]. The sequence a_0, a_1, \dots, a_n is called a Pólya frequency sequence if all minors of the lower triangular matrix $(a_{i-j})_{i,j=0}^n$ have nonnegative determinants, where a_k is set to be zero if $k < 0$. The Aissen-Schoenberg-Whitney theorem [1] states that $f(x)$ is real-rooted if and only if the sequence a_0, a_1, \dots, a_n is a Pólya frequency sequence. Another characterization from the probabilistic point of view can be found in Schoenberg [14].

Suppose that $f(x)$ has real roots $s_1 \leq \cdots \leq s_n$. Let $g(x)$ be a polynomial of degree m with real roots $t_1 \leq \cdots \leq t_m$. Following Wagner [17], we say that $g(x)$ interlaces $f(x)$, denoted $g(x) \preceq f(x)$, if $m = n - 1$ and

$$s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq t_{n-2} \leq s_{n-1} \leq t_{n-1} \leq s_n.$$

If all the above inequalities are strict, we say that $g(x)$ strictly interlaces $f(x)$, denoted $g(x) \prec f(x)$. Let $\{f_n(x)\}_{n \geq 0}$ be a sequence of polynomials with positive leading coefficients. We say that $\{f_n(x)\}_{n \geq 0}$ (or simply $f_n(x)$ if no confusion arises) forms a Sturm sequence (or, satisfies the Sturm property) if each $f_n(x)$ is real-rooted and

$$f_0(x) \prec f_1(x) \prec f_2(x) \prec \cdots.$$

Liu and Wang [12] gave some criteria for the Sturm property of polynomials, from which they derived the real-rootedness of some combinatorial polynomials, such as the Eulerian polynomials and the matching polynomials.

In this paper, we study the Sturm property of coordinator polynomials of type B , type C , and type D . Following Ardila, Beck, Hoşten, Pfeifle, and Seashore [2], we give an overview of notion and terminology. Let \mathcal{L} be a lattice, that is, a discrete subgroup of a finite-dimensional Euclidean vector space E . The dimension of the subspace spanned by \mathcal{L} is called its rank. A lattice is said to be generated as a monoid if there exists a finite collection M of vectors such that every vector in the lattice can be expressed as a nonnegative integer linear combination of the vectors in M . Let M be a finite collection of vectors, and \mathcal{L} a lattice with rank d generated by M . Let v be a vector in \mathcal{L} . The length of v with respect to M , denoted $\ell(v)$, is defined to be the minimum sum of the coefficients among all nonnegative integer linear combinations. In other words,

$$\ell(v) = \min \left\{ \sum_{i=1}^r c_i \mid v = \sum_{i=1}^r c_i a_i, c_i \geq 0, a_i \in M \right\}.$$

Let $S(k)$ be the number of vectors of length k in \mathcal{L} . Benson [5] proved that

$$\sum_{k \geq 0} S(k)x^k = h(x)(1-x)^{-d}$$

is a rational function, where $h(x)$ is a polynomial of degree less than or equal to d , called the coordinator polynomial with respect to M .

We concern ourselves with the classical root lattices as \mathcal{L} . Let e_i denote the vector in E with the i th entry one and all other entries zero. Here the space E is taken to be \mathbb{R}^{n+1} for the root lattice A_n , and \mathbb{R}^n for the root lattices B_n , C_n , and D_n . The root lattices can be defined to be generated as monoids respectively by

$$\begin{aligned} M_{A_n} &= \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}, \\ M_{B_n} &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}, \\ M_{C_n} &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}, \\ M_{D_n} &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}. \end{aligned}$$

We use h_{A_n} (resp., h_{B_n} , h_{C_n} , h_{D_n}) to denote the coordinator polynomial of type A_n (resp., B_n , C_n , D_n). Conway and Sloane [9] established the explicit expression

$$h_{A_n}(x) = \sum_{k=0}^n \binom{n}{k}^2 x^k \quad (1.1)$$

for coordinator polynomials of type A . They were also called the Narayana polynomials of type B by Chen, Tang, Wang and Yang [8]. In fact, these polynomials appeared as the rank generating function of the lattice of noncrossing partitions of type B on the set $\{1, 2, \dots, n\}$, see Simion [15]. Together with Colin Mallows, Conway and Sloane conjectured that

$$h_{D_n}(x) = \frac{(1 + \sqrt{x})^{2n} + (1 - \sqrt{x})^{2n}}{2} - 2nx(1 + x)^{n-2}. \quad (1.2)$$

Baake and Grimm [3] pointed out that by using the methods outlined in [9], it can be deduced that type C coordinator polynomials have the expression

$$h_{C_n}(x) = \sum_{k=0}^n \binom{2n}{2k} x^k. \quad (1.3)$$

They also conjectured that

$$h_{B_n}(x) = \sum_{k=0}^n \binom{2n+1}{2k} x^k - 2nx(1 + x)^{n-1}. \quad (1.4)$$

Bacher, de la Harpe and Venkov [4] rederived (1.1) and proved the formulas (1.2), (1.3) and (1.4). Recently, Ardila et al. [2] gave alternative proofs for (1.1), (1.2) and (1.3) by computing the f -vectors of a unimodular triangulation of the corresponding root polytope.

Problems on the real-rootedness of coordinator polynomials have received much attention. As pointed out by Conway and Sloane [9], coordinator polynomials of type A can be expressed as

$$h_{A_n}(x) = (1 - x)^n L_n\left(\frac{1 + x}{1 - x}\right),$$

where $L_n(x)$ denotes the n th Legendre polynomial. Since Legendre polynomials are orthogonal (which satisfy the Sturm property), the following result is immediate.

Proposition 1.1 *The coordinator polynomials of type A form a Sturm sequence.*

Using the technique due to Liu and Wang [12] of proving the real-rootedness of sequences of polynomials, Liang and Yang [11] gave another proof of the above proposition based on a recurrence relation satisfied by $h_{A_n}(x)$. Using the same method, they proved that both the coordinator polynomials of type C and the polynomials

$$h_{B'_n}(x) = \sum_{k=0}^n \binom{2n+1}{2k} x^k \quad (1.5)$$

are real-rooted.

However, it seems that there do not exist similar recurrence relations for $h_{B_n}(x)$ and $h_{D_n}(x)$ to which Liu and Wang's criterion can be applied. In fact, $h_{B_n}(x)$ is not real-rooted in general, as shown in Section 3. For the coordinator polynomials of type D , Liang and Yang conjectured that these polynomials form a Sturm sequence. The main objective of this paper is to prove this conjecture.

Our approach is to transform the polynomial $h_{D_n}(x)$ into a trigonometric function, say, $g(\theta)$. It is clear that all the real roots of $h_{D_n}(x)$ lie on the negative axis. Using this substitution, we consider the zeros of $g(\theta)$ on the new interval $[0, \frac{\pi}{2}]$. It turns out that we can determine the signs of $g(\theta)$ at a sequence of precise points, from which we obtain the Sturm property of $h_{D_n}(x)$.

This paper is organized as follows. In the next section, we prove the Sturm property of $h_{D_n}(x)$. In Section 3, we give a trigonometric approach to the Sturm property of $h_{C_n}(x)$ and $h_{B'_n}(x)$, and discuss the real-rootedness of $h_{B_n}(x)$ as well.

2 The main result

Recall that type D coordinator polynomials $h_{D_n}(x)$ are defined by the formula (1.2). Here is our main result.

Theorem 2.1 *For $n \geq 2$ the polynomials $h_{D_n}(x)$ form a Sturm sequence.*

In this section, we shall give a proof of Theorem 2.1 with the aid of trigonometric substitutions. To this end, we first prove the real-rootedness of $h_{D_n}(x)$, and then the interlacing property. First of all, we give a lemma which will be needed later.

Lemma 2.2 *Let a, b, c be positive integers. Suppose that $a = \alpha\beta$ where $\alpha, \beta > 0$. Let $f(y) = (c - y^\alpha)^\beta y^b$. If $0 \leq y^\alpha \leq c$, then we have*

$$f(y)^\alpha \leq a^a b^b \left(\frac{c}{a+b} \right)^{a+b},$$

where the equality holds if and only if $(a+b)y^\alpha = bc$. In particular, for $|y| \leq 1$ we have

$$(1 - y^2)^2 y^{2b} \leq \frac{4b^b}{(b+2)^{b+2}}. \quad (2.1)$$

Proof. By the arithmetic-geometric mean inequality, we have

$$f(y)^\alpha = \frac{(ay^\alpha)^b (bc - by^\alpha)^a}{a^b b^a} \leq \frac{1}{a^b b^a} \left(\frac{abc}{a+b} \right)^{a+b} = a^a b^b \left(\frac{c}{a+b} \right)^{a+b}.$$

By taking $c = \beta = 1$, $\alpha = 2$ and $f(y) = (1 - y^2)y^b$, we arrive at the inequality (2.1). This completes the proof. ■

Proposition 2.3 For $n \geq 2$, all roots of the polynomial $h_{D_n}(x)$ are real.

Proof. For convenience, we write $f_n(x) = 2h_{D_n}(x)$, namely

$$\begin{aligned} f_n(x) &= (1 + \sqrt{x})^{2n} + (1 - \sqrt{x})^{2n} - 4nx(1+x)^{n-2} \\ &= 2 \sum_k \binom{2n}{2k} x^k - 4nx(1+x)^{n-2}. \end{aligned} \quad (2.2)$$

Let $y > 0$ and $x = -y^2$. We have

$$\begin{aligned} 2 \sum_k \binom{2n}{2k} x^k &= 2 \sum_k \binom{2n}{2k} (yi)^{2k} = \sum_k \binom{2n}{k} (yi)^k + \sum_k \binom{2n}{k} (-yi)^k \\ &= (1 + yi)^{2n} + (1 - yi)^{2n}. \end{aligned}$$

Therefore

$$f_n(-y^2) = (1 + yi)^{2n} + (1 - yi)^{2n} + 4ny^2(1 - y^2)^{n-2}. \quad (2.3)$$

We remark that the validity of (2.3) can not be deduced from (2.2) directly because the simplification $\sqrt{-y^2} = yi$ is not clear according to the definition of a square root of a negative number.

Let $\theta \in (0, \pi/2)$ and $y = \tan \theta$. Note that $1 + yi = \sqrt{1 + y^2}e^{i\theta}$. Routine computations yield that

$$f_n(-y^2) = 2(1 + y^2)^n \left(\cos 2n\theta + \frac{n}{2} \sin^2 2\theta \cos^{n-2} 2\theta \right).$$

Let $\phi = 2\theta$. Then $\phi \in (0, \pi)$ and

$$f_n(-y^2) = 2(1 + y^2)^n g_n(\phi),$$

where

$$g_n(\phi) = \cos n\phi + \frac{n}{2} \sin^2 \phi \cos^{n-2} \phi. \quad (2.4)$$

Now it suffices to prove that $g_n(\phi)$ has n distinct roots lying in the interval $(0, \pi)$. Let

$$h_n(\phi) = \frac{n}{2} \sin^2 \phi \cos^{n-2} \phi.$$

By (2.1), we have $|h_n(\phi)| < 1$. Let j be an integer. Then

$$g_n\left(\frac{j\pi}{n}\right) = (-1)^j + h_n\left(\frac{j\pi}{n}\right).$$

Thus the sign of $g_n(\frac{j\pi}{n})$ is the same as that of $(-1)^j$, i.e.,

$$(-1)^j g_n\left(\frac{j\pi}{n}\right) > 0. \quad (2.5)$$

By the continuity of $g_n(\phi)$, it has roots $\phi_0, \phi_1, \dots, \phi_{n-1}$ such that

$$0 < \phi_0 < \frac{\pi}{n} < \phi_1 < \frac{2\pi}{n} < \phi_2 < \frac{3\pi}{n} < \dots < \frac{(n-1)\pi}{n} < \phi_{n-1} < \pi. \quad (2.6)$$

Hence the function $f_n(x)$ has n distinct negative roots $\{-\tan^2 \frac{\phi_j}{2}\}_{j=0}^{n-1}$. This completes the proof. \blacksquare

Now we can suppose that the function $g_{n+1}(\phi)$ has roots $\psi_0, \psi_1, \dots, \psi_n$ in the interval $(0, \pi)$.

Proof of Theorem 2.1. Since the expression $x = -\tan^2 \frac{\phi}{2}$ of the roots of $f_n(x)$ is monotone in ϕ , it suffices to show that

$$0 < \psi_0 < \phi_0 < \psi_1 < \phi_1 < \dots < \psi_{n-1} < \phi_{n-1} < \psi_n < \pi.$$

Here comes a key observation. Since $g_{n+1}(0) = 1 > 0$ and $g_{n+1}(\pi) = (-1)^{n+1}$, the above inequalities hold if and only if for all $0 \leq j \leq n-1$,

$$(-1)^j g_{n+1}(\phi_j) < 0. \quad (2.7)$$

By (2.4), we have $g_n(\phi) = (-1)^n g_n(\pi - \phi)$ for any ϕ . It follows that

$$\phi_j = \pi - \phi_{n-1-j} \quad (2.8)$$

for all $0 \leq j \leq n-1$. Since

$$(-1)^{n-1-j} g_{n+1}(\phi_{n-1-j}) = (-1)^{n-1-j} (-1)^{n+1} g_{n+1}(\pi - \phi_{n-1-j}) = (-1)^j g_{n+1}(\phi_j),$$

the inequality system (2.7) can be reduced to its first half inequalities, in other words, those for $j \leq \lfloor \frac{n-1}{2} \rfloor$. On the other hand, it is easy to find that

$$g_{n+1}(\theta) = g_n(\theta) \cos \theta + F_n(\theta) \sin \theta,$$

where

$$F_n(\theta) = \frac{1}{2} \sin \theta \cos^{n-1} \theta - \sin n\theta. \quad (2.9)$$

Therefore $g_{n+1}(\phi_j) = F_n(\phi_j) \sin \phi_j$. Let $j \leq \lfloor \frac{n-1}{2} \rfloor$. By (2.6) we have

$$\phi_j \in \left(\frac{j\pi}{n}, \frac{(j+1)\pi}{n} \right). \quad (2.10)$$

Together with the relation (2.8), we find that $\phi_j \in (0, \frac{\pi}{2}]$. So $\sin \phi_j > 0$ and consequently (2.7) can be recast as

$$(-1)^j F_n(\phi_j) < 0, \quad (2.11)$$

for all $0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$. If $\phi_j = \frac{\pi}{2}$, then n is odd and $j = \frac{n-1}{2}$ by (2.8). In this case, the inequality (2.11) holds trivially. If $n = 2$, then $j = 0$, the above inequality reduces

to $\sin 2\phi_0 > 0$, which holds since $2\phi_0 \in (0, \pi)$. Below we can suppose that $n \geq 3$, $\phi_j \in (0, \frac{\pi}{2})$, and correspondingly $0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$.

By the expression (2.9), we observe that it suffices to prove

$$\sin \phi_j < (-1)^j \sin n\phi_j. \quad (2.12)$$

Note that for any $\alpha \in (0, \frac{\pi}{2}]$ and $\beta \in (0, \pi)$, there holds

$$\sin \alpha < \sin \beta \quad \Leftrightarrow \quad \alpha < \beta < \pi - \alpha.$$

Since $(-1)^j \sin n\phi_j = \sin(n\phi_j - j\pi)$ where $n\phi_j - j\pi \in (0, \pi)$, we find that the inequality (2.12) is equivalent to

$$\frac{j\pi}{n-1} < \phi_j < \frac{(j+1)\pi}{n+1}. \quad (2.13)$$

Below we prove the inequality (2.13). Note that $0 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1$, we have

$$\frac{j\pi}{n} \leq \frac{j\pi}{n-1} < \frac{(j+1)\pi}{n+1} < \frac{(j+1)\pi}{n}.$$

Consider the continuous function $G_n(\phi) = (-1)^j g_n(\phi)$. The result (2.5) implies that

$$G_n\left(\frac{j\pi}{n}\right) > 0 \quad \text{and} \quad G_n\left(\frac{(j+1)\pi}{n}\right) < 0.$$

Since ϕ_j is the unique root in the interval $(\frac{j\pi}{n}, \frac{(j+1)\pi}{n})$, the inequality (2.13) holds if and only if both

$$G_n\left(\frac{j\pi}{n-1}\right) > 0 \quad (2.14)$$

and

$$G_n\left(\frac{(j+1)\pi}{n+1}\right) < 0. \quad (2.15)$$

Now we are going to verify (2.14) and (2.15). Note that

$$\cos \frac{jn\pi}{n-1} = (-1)^j \cos \frac{j\pi}{n-1}.$$

Let $y = \cos \frac{j\pi}{n-1}$. Then $y \in (0, 1]$ and

$$G_n\left(\frac{j\pi}{n-1}\right) = y \left(1 + \frac{(-1)^j n}{2} (1 - y^2) y^{n-3}\right).$$

If $n = 3$, then $j = 0$ and (2.14) follows from (2.5). For $n \geq 4$, by (2.1) we have

$$(1 - y^2)^2 y^{2n-6} \leq \frac{4(n-3)^{n-3}}{(n-1)^{n-1}} \leq \frac{4(n-3)}{(n-1)^3} < \frac{4}{n^2}. \quad (2.16)$$

This implies (2.14) immediately. It remains to prove (2.15). Observe that

$$\cos \frac{(j+1)n\pi}{n+1} = (-1)^{j+1} \cos \frac{(j+1)\pi}{n+1}.$$

Let $z = \cos \frac{(j+1)\pi}{n+1}$. Then $z \in (0, 1)$ and

$$G_n\left(\frac{(j+1)\pi}{n+1}\right) = z\left(-1 + \frac{(-1)^j n}{2}(1 - z^2)z^{n-3}\right). \quad (2.17)$$

If $n = 3$, then $j = 0$, $z = 1/\sqrt{2}$, and

$$G_n\left(\frac{(j+1)\pi}{n+1}\right) = \frac{1}{\sqrt{2}}\left[-1 + \frac{3}{4 \cdot (\sqrt{2})^{n-3}}\right] < 0.$$

For $n \geq 4$, replacing y by z in (2.16) we obtain that

$$(1 - z^2)z^{n-3} < \frac{2}{n}.$$

By (2.17), we get (2.15). This completes the proof. ■

3 The coordinator polynomials of other types

In this section, we deal with coordinator polynomials of type B and type C , as well as the polynomial $h_{B'_n}(x)$ defined by (1.5).

Proposition 3.1 *The coordinator polynomials of type C form a Sturm sequence, and so do the polynomials $h_{B'_n}(x)$.*

Proof. By the expressions (1.3) and (1.5), the polynomials $h_{C_n}(x)$ and $h_{B'_n}(x)$ can be written uniformly as

$$U_m(x) = \sum_k \binom{m}{2k} x^k = \frac{(1 + \sqrt{x})^m + (1 - \sqrt{x})^m}{2},$$

where $m \geq 2$. In this notation, we have

$$U_m(x) = \begin{cases} h_{C_n}(x), & \text{if } m = 2n; \\ h_{B'_n}(x), & \text{if } m = 2n + 1. \end{cases}$$

It is a polynomial of degree $\lfloor \frac{m}{2} \rfloor$. Now we use the same trigonometric substitution as in the proof of Theorem 2.1. Let $\theta \in (0, \pi/2)$ and $y = \tan \theta$. Then $y > 0$ and

$$U_m(-y^2) = (1 + y^2)^{m/2} \cos m\theta.$$

So $U_m(x)$ has the following $\lfloor \frac{m}{2} \rfloor$ distinct real roots

$$-\tan^2 \frac{(2k+1)\pi}{2m}, \quad k = 0, 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

Since the degree of $U_m(x)$ is exactly $\lfloor \frac{m}{2} \rfloor$, we find that the above roots consist of a complete list of roots of $U_m(x)$. The Sturm property follows immediately. This completes the proof. \blacksquare

For coordinator polynomials of type B , we observe that the polynomial

$$\begin{aligned} h_{B_{16}}(x) = & 2x^{16} + 9952x^{15} + 467952x^{14} + 8514976x^{13} + 77046840x^{12} + 386881248x^{11} \\ & + 1146012560x^{10} + 2073904800x^9 + 2333194380x^8 + 1637298080x^7 \\ & + 709442448x^6 + 185034720x^5 + 27739192x^4 + 2208416x^3 \\ & + 80880x^2 + 992x + 2 \end{aligned}$$

has exactly 14 distinct real roots and 2 non-real roots, which says that they do not form a Sturm sequence.

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